Analysis of Sequential Frame Synchronizers in Gaussian Noise Channels

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ABSTRACT
We present a framework for the analysis of frame synchronization based on Synchronization Words (SWs), where the detection is based on the common sequential algorithm: the received samples are observed over a window of length equal to the SW; over this window a metric (e.g., correlation) is computed; a SW is declared if the computed metric is greater than a proper threshold, otherwise the observation window is time-shifted of one sample. We assume a Gaussian channel, antipodal signalling and coherent detection, where soft values are provided to the frame synchronizer. We state the problem starting from the hypothesis testing theory, deriving the optimum metric (optimum likelihood ratio test (LRT)) according to the Neyman-Pearson lemma. When the data distribution is unknown, we design a simple and effective test based on the Generalized LRT (GLRT). We also analyze the performance of the commonly used correlation metric, both in the "hard" and "soft" version. We show that synchronization by correlation can be greatly improved by the LRT and GLRT metrics, and also that, among correlation based tests, sometimes hard correlation is better than soft correlation. The obtained closed form expressions allow the derivation of the receiver operating characteristic (ROC) curves for the LRT and GLRT synchronizers, showing a remarkable gain with respect to synchronization based on correlation metric. The effect on the performance of non-equally distributed data is also shown.

Keywords: Frame synchronization, hypothesis testing, detection, LRT, GLRT, synchronization patterns

1. INTRODUCTION
Frame synchronization is a critical issue in communication [1–4]. An example is video transmission: if synchronization markers (sync words, SW’s, or start codes) are inserted in the bitstream, synchronization can be re-gained after errors have occurred. This is the case, e.g., of the MPEG-4 video stream [5,6], where SW’s are aperiodically embedded in the bitstream. The SW length and detection strategy should be chosen in this case according to channel conditions and source characteristics. The quality degradation due to imperfect synchronization depends on the probability of missing the SW (missed detection) and on the probability of false start code detection (false alarm). It is thus of importance the search for synchronization algorithms minimizing these probabilities.

The problem of frame synchronization and of the relevant search algorithms has been afforded in [2], where also the design of good synchronization codes is treated.

In the case of periodically embedded synchronization words, i.e. the case of fixed length frames of \( N_D \) data symbols delimited by sync words of length \( N \), frame synchronization can be performed through the search of the maximum of a metric in a window of \( N_f = N_D + N \) symbols. More precisely, for each of the possible \( N_D + N \) positions of the SW in the observation window, a metric is evaluated over \( N \) received symbols. The position of the SW is chosen as the one corresponding to the maximum evaluated metric. In the binary symmetric channel (BSC) case the optimal metric is simply the correlation between the observed \( N \)-symbols sequence and the SW pattern [1]. In additive white Gaussian noise (AWGN) channels with soft outputs this is not true, as shown in [7], where the optimum metric for frame synchronization in the case of periodically embedded sync words is derived. However, regardless of its sub-optimality, detection through correlation has become a common engineering practice even when soft values are available [3] [4].

Several studies have been performed considering periodically embedded sync words. The performance evaluation for binary symmetric channels has been studied in [1], where also synchronization sequences with good
aperiodic autocorrelation properties have been identified. In [3] a basic theory of frame synchronization is also presented and considerations on the marker design are made. A frame sync acquisition algorithm, based on the comparison of the correlation metric with a threshold, is considered for the Gaussian channel with coherent demodulation. Also in [3], following [7], a "deferred decision frame sync acquisition" based on correlation is proposed, in order to obtain a compromise between performance and simplicity. In [8] a union lower bound on synchronization probability for this acquisition rule on AWGN channels was determined. In [9] a performance evaluation through simulation of various metrics based on the search of the maximum in a fixed length window has been presented.

The aperiodically embedded case has been less considered in literature. The expected duration of a search for a fixed pattern in a semi-infinite stream of random data is addressed in [10], where also its dependency on the structure of the pattern is analyzed. The extension of the optimal frame sync approach [7] to non binary modulation schemes and to frames of known but not necessary constant lengths can be found in [11,12], where the search is based on the maximization of a proper metric over a fixed window; in the same works, numerical results describing the performance of the synchronizer are provided through simulation.

However, there is a lack of studies on optimal frame sync algorithms also valid for the general case of synchronization of frames of unknown, variable lengths.

In this paper, the case of synchronization patterns aperiodically embedded in the bitstream is considered and afforded in the hypothesis testing framework [13], [14]. We assume we do not have a-priori information on the frame lengths. In this case, an acquisition algorithm based on a step-by-step comparison of a proper metric with a threshold is considered and the detection metrics for AWGN channels based on the likelihood ratio test and its generalized version are derived. An analytical performance evaluation of such tests is given. case In order to benchmark the obtained results, the analysis of frame synchronization through the correlation metric is also reported, both for a BSC and for an AWGN channel model, and represented in terms of receiver operating characteristic (ROC) curves. Note that the derived synchronizer may be also used, as particular case, for fixed length frames.

It will be shown that the new metrics, derived according to LRT and GLRT, provide large gains with respect to those based on correlation, both in the "soft" and "hard" versions

2. PROBLEM STATEMENT

We consider binary antipodal signalling, so that the $i$-th transmitted bit $b_i \in \{0,1\}$ gives rise, after binary antipodal modulation, transmission through the AWGN channel, matched filter reception and perfect sampling, to a sample $r_i = (-1)^{b_i} + n_i$, where $n_i$ are independent, identically distributed (i.i.d.) Gaussian random variables (r.v.’s), with zero mean and variance $\sigma^2$. This model is also valid for BPSK systems, for which the signal-to-noise ratio is $E_s/N_0 = 1/(2\sigma^2)$, where $E_s$ is the energy per symbol and $N_0$ is the one-sided thermal noise power spectral density.

We assume that an $N$ binary symbols sync word, $(c_1,\ldots,c_N)$, is aperiodically inserted in the data stream, composed of symbols $d_i \in \{-1,+1\}$ that are i.i.d. r.v.’s with equiprobable $-1$ and $+1$. More generally, we will consider in the generalized likelihood test case also the case of non-equally distributed data.

Each SW symbol $c_i$ is either $-1$ or $+1$.

The acquisition algorithm we consider is as follows: starting from a position $k$, the synchronizer observes a vector of $N$ subsequent samples; based on a suitable metric evaluated from this vector it decides if the SW is in position $k$; if not, it moves to position $k + 1$, repeating the steps until the sync word is detected.

In this paper, we afford the problem of deciding at each position $k$ of the bitstream whether a sync word is present or not. The relation between this problem and other performance indicators such as, e.g., the probability of correct acquisition in one pass, is addressed in [3,10].

We assume in the following that the statistical properties of the metric do not depend on the position in the bitstream. We thus avoid considering the effect of the aperiodic autocorrelation around SW’s. We have evaluated

*We refer to soft values as the real valued samples at the AWGN channel output and to hard values as their binary quantized version.
in fact that, if the SW is properly chosen as a sequence with optimized aperiodic autocorrelation property (e.g. Barker sequences [1], or those in [3,15,16]) its symbols should mimic random data, with, moreover, the additional property that some configurations can be avoided.

We consider thus, without loosing in generality, $k = 1$ and we neglect in the design and analytical performance evaluation of the frame synchronizer the case of ”mixed data”, i.e., when both data and SW symbols are present in the metric evaluation window. However, simulation results will be given in Section 6 for the case of ”mixed data” to support this approximation. In particular we will show that if the SW is properly designed the case of purely random data represents generally a worst case in terms of probability of false sync word detection respect to the ”mixed data” case.

We study the problem through the statistical theory of hypothesis testing. After observing $N$ subsequent samples, the synchronizer must choose between two possible situations

\begin{equation}
\begin{aligned}
H_0 &: r_i = d_i + n_i, \quad i = 1, \ldots, N \\
H_1 &: r_i = c_i + n_i, \quad i = 1, \ldots, N,
\end{aligned}
\end{equation}

the first hypothesis representing the case where there is no sync word, the second corresponding to the case the sync word is present. Decisions are indicated by $D_0, D_1$, corresponding to the ”true” hypotheses $H_0, H_1$, respectively.

The probability of emulation, $P_{EM}$, or false start code detection †, of choosing hypothesis $H_1$ when $H_0$ is true is

\begin{equation}
P_{EM} = \Pr(D_1|H_0).
\end{equation}

The probability of missed detection, $P_{MD}$, of choosing $H_0$ when $H_1$ is true, is

\begin{equation}
P_{MD} = \Pr(D_0|H_1)
\end{equation}

and the probability of correct detection is

\begin{equation}
P_D = 1 - P_{MD}.
\end{equation}

3. SEQUENTIAL FRAME SYNCHRONIZATION TESTS

3.1. Derivation of the LRT

Assuming AWGN and soft values available, the likelihood ratio test (LRT) [14] is considered for the derivation of the optimal detection metric. In the following we use capital letters to indicate random variables and bold for vectors.

By indicating with $R = (R_1, \ldots, R_N)$ the r.v. corresponding to the vector $r = (r_1, \ldots, r_N)$ of received samples, the LRT is

\begin{equation}
\Lambda'(r) = \begin{cases} 
\frac{f_{R|H_0}(r|H_0)}{f_{R|H_1}(r|H_1)} & D_0 \gtrless \lambda' \\
\end{cases}
\end{equation}

where $f_{R|H_j}(r|H_j)$ is the p.d.f. of $R$ under hypothesis $H_j$, $j = 0, 1$, and $\lambda'$ is the selected threshold. Thus, according to the test, $\Lambda'(r) < \lambda'$ corresponds to the decision $D_1$, i.e. we decide we are in presence of a sync word; otherwise, the decision is $D_0$. In [17] we derived the Log-Likelihood Ratio Test (LLRT) as

\begin{equation}
\Lambda = \Lambda(r) = \sum_{i=1}^{N} \ln \left(1 + e^{-2 ric_i \sigma^2} \right) \begin{cases} 
D_0 \gtrless \lambda \\
\end{cases}
\end{equation}

†Here the false start code detection is due to the case where random data plus noise is interpreted as a SW. This can occur either in the case data symbols are coincident with the SW pattern or, due to noise, even if data symbols are different from the SW pattern.
We thus decide $D_1$ (the start code is present) if $\Lambda(r) < \lambda$, $D_0$ otherwise. The threshold $\lambda$ is chosen according to the Neyman-Pearson criterion, i.e. by fixing the maximum tolerable probability of false alarm (emulation). We should note that the obtained metric depends on channel conditions, expressed in (6) in terms of $\sigma^2$ and thus requires the instantaneous knowledge of the signal to noise ratio in order to perform the detection process. Furthermore, the evaluation of a non-linear function is required. The LRT based synchronizer is depicted in Fig. 2, where $\phi(x) = \ln(1 + e^x)$.

3.2. Derivation of the GLRT

In order to benchmark the performance of the previously derived metric, we here present another possible test that will be shown to have some advantages in terms of implementation complexity.

In particular, in [18] we started affording frame synchronization for those situations where we cannot make any assumption about the probability distribution of data symbols $d_i$. In this case we may resort to the generalized likelihood ratio test (GLRT) [14] for the design of the detection strategy, since designing the likelihood ratio test requires the knowledge of the data distribution. The GLRT approach leads in principle to a two steps procedure: first, we estimate the unknown data vector $d = (d_1, \ldots, d_N)$ under hypothesis $H_0$; then, we use the estimate $\hat{d} = (\hat{d}_1, \ldots, \hat{d}_N)$ as if it were the correct value of the transmitted data. The GLRT is

$$
\Lambda'_g(r) = \frac{\int_{R|H_1}(r|H_1)}{\int_{R|H_0,d}(r|H_0,d)} \frac{D_1}{D_0} \lambda',
$$

(7)

where $\lambda'$ is the selected threshold. In our case we use the maximum likelihood estimate for each of the $d_i$, that is clearly $\hat{d}_i = \text{sign}(r_i)$, for $i = 1, \ldots, N$.

Starting from (7) the GLR test becomes [18]

$$
\Lambda_g = \Lambda_g(r) = \sum_{i=1}^{N} (|r_i| - r_i c_i) \frac{D_0}{D_1} \lambda,
$$

(8)

where $\lambda \propto \ln(\lambda')$. Note that since $|c_i| = 1$, the quantity to be evaluated in test (8) can be interpreted as twice the sum of the absolute value of the samples $r_i$ whose signs differ from that of the SW symbols $c_i$. In other words, the GLRT based synchronizer can just discard those samples whose signs are equal to that of the SW, and compare the sum of the remaining (absolute values) with a threshold.

We also observe that this metric does not depend on the signal to noise ratio and it may thus be used also when information on the channel conditions is not available.

Thus, from an implementation point of view the synchronizer based on the GLRT (8) is simpler than that based on the optimum LRT (6). We will see later that, despite its simplicity, the GLRT performs very similarly to LRT except for very low signal to noise ratios. Indeed, since the function $\ln(1 + e^x)$ may be approximated by 0 for $x \ll 0$ and by $x$ for $x \gg 0$, it is simple to verify that the LRT in (6) tends to the GLRT for increasing signal to noise ratios. Moreover, the GLRT does not need any assumption about data distribution (i.e. $Pr\{d_i = +1\}$), whereas, in order to design the optimal LRT, this distribution is required at the synchronizer. From this point of view it is worthwhile to remark that the LRT in (6) looses its optimality when applied to unbalanced data symbols.

4. PERFORMANCE EVALUATION OF SEQUENTIAL TESTS ON AWGN

In this section we provide the performance evaluation of the presented tests in terms of probability of false start code detection, or emulation, and of probability of missed detection.

First, let us observe that the probability of false alarm or emulation, $P_{EM}$, can be evaluated as

$$
P_{EM} = Pr\{\Lambda < \lambda|H_0\} = F_{\Lambda|H_0}(\lambda|H_0),
$$

(9)
where \( \Lambda \in \{ \Lambda, \Lambda_g \} \) represents the considered metric, \( \lambda \) is the selected threshold in (6) or (8), and \( F_{\Lambda,|H_l}(\cdot) \) is the cumulative distribution function (c.d.f.) of \( \Lambda \) under hypothesis \( H_l \), \( l = 0, 1 \). Similarly, the probability of missed detection, \( P_{MD} \), is given by

\[
P_{MD} = \Pr\{\Lambda > \lambda | H_1\} = 1 - F_{\Lambda,|H_1}(\lambda | H_1).
\]  

Thus, the performance analysis of the synchronizer passes through the evaluation of the c.d.f.’s of the involved metric.

4.1. Performance analysis of LRT

In order to evaluate the performance of the optimal test we need to evaluate the statistical distribution of the r.v. \( \Lambda(R) \) as defined in (6), for the two hypotheses \( H_0, H_1 \).

To this aim we first study the distribution of the auxiliary r.v.’s

\[
W_i = 1 + \exp\left\{-\frac{2R_i c_i}{\sigma^2}\right\} > 1
\]

and

\[
Z_i = \ln(W_i),
\]

so that the statistical analysis of the LLR in (6) is derived by studying the r.v.

\[
\Lambda(R) = \sum_{i=1}^{N} \ln W_i = \sum_{i=1}^{N} Z_i.
\]

Under hypothesis \( H_l \), \( l = 0, 1 \), the cumulative distribution function of \( W_i \) is

\[
F_{W_i|H_l}(w | H_l) = \Pr\{W_i \leq w | H_l\}
= \Pr\left\{1 + e^{-\frac{2R_i c_i}{\sigma^2}} \leq w | H_l\right\}.
\]

Considering, without loss of generality\(^1\), \( c_i = 1 \), we have

\[
F_{W_i|H_l}(w | H_l) = \Pr\{R_i \geq -\sigma^2 \ln \sqrt{w - 1} | H_l\}
= 1 - F_{R_i|H_l}\left(-\sigma^2 \ln \sqrt{w - 1} | H_l\right),
\]

where \( F_{R_i|H_l}(\cdot) \) is the c.d.f. of \( R_i \) given \( H_l \).

Using the basic theory for the transformation of r.v.’s, we also have from (12) the relation

\[
F_{Z_i|H_l}(z | H_l) = F_{W_i|H_l}(e^z | H_l),
\]

with \( z \geq 0 \).

4.1.1. LLR statistical distribution under \( H_0 \)

Since under hypothesis \( H_0 \) the p.d.f. of the r.v. \( R_i \) is the sum of two Gaussian p.d.f.’s centered in \( \pm 1 \) multiplied by \( 1/2 \), applying (14) we have

\[
1 - F_{W_i|H_0}(w | H_0) = \frac{1}{4} \text{erfc}\left(\frac{-1 + \sigma^2 \ln \sqrt{w - 1}}{\sqrt{2}\sigma}\right)
+ \frac{1}{4} \text{erfc}\left(\frac{1 + \sigma^2 \ln \sqrt{w - 1}}{\sqrt{2}\sigma}\right)
\]

where

\[
\text{erfc}(x) \triangleq \frac{2}{\sqrt\pi} \int_{x}^{\infty} e^{-t^2} dt
\]

\(^1\)Under hypothesis \( H_0 \) this is true only for equiprobable data symbols, that is what we assumed.
is the complementary Gaussian error function.

Hence, for the r.v. $Z_i$ we have

$$F_{Z_i|\mathcal{H}_0}(z|\mathcal{H}_0) = \frac{1}{4} \text{erfc} \left( \frac{-1 - \sigma^2 \ln \sqrt{e^z - 1}}{\sqrt{2\sigma}} \right) + \frac{1}{4} \text{erfc} \left( \frac{1 - \sigma^2 \ln \sqrt{e^z - 1}}{\sqrt{2\sigma}} \right).$$

(17)

Deriving with respect to $z$ we obtain the p.d.f.

$$f_{Z_i|\mathcal{H}_0}(z|\mathcal{H}_0) = \frac{\sigma}{4\sqrt{2\pi}} \frac{e^z}{\sqrt{e^z - 1}} \left[ e^{-\frac{1}{2\sigma^2} (1+\sigma^2 \ln \sqrt{\frac{1}{e^z} - 1})^2} + e^{-\frac{1}{2\sigma^2} (1-\sigma^2 \ln \sqrt{\frac{1}{e^z} - 1})^2} \right].$$

(18)

We now recall that the characteristic function (ch.f.) of the sum of independent random variables is the product of the ch.f.'s of the single terms. Since we are interested in the distribution of $\Lambda(\mathcal{R})$ defined in (6) that is the sum of $N$ independent r.v.'s each with p.d.f. (18), it is convenient to evaluate the ch.f. of the r.v. $Z_i$, here defined as

$$\Phi_{Z_i|\mathcal{H}_0}(\nu) \triangleq \mathbb{E}[e^{j2\pi \nu Z_i}] = \int_{-\infty}^{\infty} f_{Z_i|\mathcal{H}_0}(z)e^{j2\pi \nu z} dz,$$

(19)

being $j = \sqrt{-1}$ the imaginary unit.

Starting from the ch.f., both the p.d.f. and c.d.f. can be easily evaluated from (18) by using standard FFT techniques. For example

$$f_{\Lambda|\mathcal{H}_0}(\lambda|\mathcal{H}_0) = \mathcal{F} \left\{ \left[ \Phi_{Z_i|\mathcal{H}_0}(\nu) \right]^N \right\}$$

(20)

where we indicated with $\mathcal{F}$ the Fourier transform. Similarly the c.d.f. can be obtained as

$$F_{\Lambda|\mathcal{H}_0}(\lambda|\mathcal{H}_0) = \int_{-\infty}^{\infty} \Phi_{Z_i|\mathcal{H}_0}(\nu) \left[ \frac{1 - e^{-j2\pi \nu \lambda}}{j2\pi \nu} \right] d\nu.$$

(21)

This is then used in (9) to get the probability of false alarm.

4.1.2. LLR statistical distribution under $\mathcal{H}_1$

We consider now the case the sync word is present. In this situation an error is made if the sync word is not detected. We here derive the probability $P_{MD} = \Pr \{ D_0 | \mathcal{H}_1 \}$.

With the assumption $c_i = 1$, from (2) it results that $R_i|\mathcal{H}_1$ is a Gaussian r.v. with mean 1 and variance $\sigma^2$.

Consequently, with the positions made above, from (14) we get

$$F_{W_i|\mathcal{H}_1}(w|\mathcal{H}_1) = 1 - \frac{1}{2} \text{erfc} \left( \frac{1 + \sigma^2 \ln \sqrt{\frac{1}{w} - 1}}{\sqrt{2\sigma}} \right).$$

(22)

By substituting, we have

$$F_{Z_i|\mathcal{H}_1}(z|\mathcal{H}_1) = 1 - \frac{1}{2} \text{erfc} \left( \frac{1 + \sigma^2 \ln \sqrt{e^z - 1}}{\sqrt{2\sigma}} \right).$$

(23)

Deriving with respect to $z$ we obtain

$$f_{Z_i|\mathcal{H}_1}(z|\mathcal{H}_1) = \frac{\sigma}{2\sqrt{2\pi}} \frac{e^z}{\sqrt{e^z - 1}} e^{-\frac{1}{2\sigma^2} (1+\sigma^2 \ln \sqrt{\frac{1}{e^z} - 1})^2}$$

and therefore, indicating by $\Phi_{Z_i|\mathcal{H}_1}(\nu)$ the corresponding characteristic function, we get

$$f_{\Lambda|\mathcal{H}_1}(\lambda|\mathcal{H}_1) = \mathcal{F} \left\{ \left[ \Phi_{Z_i|\mathcal{H}_1}(\nu) \right]^N \right\}.$$

(25)

Again, this function and the c.d.f. can be easily evaluated from (24) by using standard FFT techniques.
4.2. Performance analysis of GLRT

We provide in this subsection a performance analysis of the GLRT synchronizer for not necessarily equiprobable data symbols. We first observe that, since \(|c_i| = 1\), we can equivalently write the metric in (8) as

\[
\Lambda_y = \sum_{i=1}^{N} |r_i c_i| - r_i c_i = \sum_{i=1}^{N} v_i,
\]  

(26)

where \(v_i = |\psi_i| - \psi_i\) with \(\psi_i = r_i c_i\). Hence,

\[
v_i = \begin{cases} 
0 & \text{if } \psi_i \geq 0, \\
-2\psi_i & \text{otherwise}. 
\end{cases}
\]

(27)

From (27), by using the rules for transformation of random variables, the p.d.f. of \(V_i\) is

\[
f_{V_i|H_i}(v) = \frac{1}{2} f_{\psi_i|H_i} \left(-\frac{v}{2}\right) u(v) + \Pr\{\psi_i \geq 0|H_i\} \delta(v)
\]

(28)

where \(u(\cdot)\) is the unitary step function and \(\delta(\cdot)\) is the Dirac delta function.

By using (28) we obtain

\[
\Phi_{V_i|H_i}(\nu) = \mathbb{E}\left[e^{j2\pi\nu V_i}\right] = \Pr\{\psi_i \geq 0|H_i\} + \frac{1}{2} \int_{0}^{\infty} f_{\psi_i|H_i} \left(-\frac{v}{2}\right) e^{j2\pi\nu v} dv.
\]

(29)

Then, the c.h.f. of \(\Lambda_y(R)\) is simply

\[
\Phi_{\Lambda_y|H_i}(\nu) = \prod_{i=1}^{N} \Phi_{V_i|H_i}(\nu).
\]

(30)

We now specialize (29) to the two hypotheses \(H_0, H_1\).

4.2.1. Case \(H_0\)

Here the sync word is not present and the metric is applied to random data. Let us denote with \(p_G(v; \mu, \sigma^2) = 1/\sqrt{2\pi\sigma^2} \exp\{-\frac{(v - \mu)^2}{2\sigma^2}\}\) the Gaussian p.d.f. with argument \(v\), mean \(\mu\) and variance \(\sigma^2\).

The conditional r.v. \(R_i|d_i\) has p.d.f. \(f_{R_i|d_i}(r) = p_G(r; d_i, \sigma^2)\). Denoting \(\Pr\{d_i = 1\} = \alpha\) and \(\Pr\{d_i = -1\} = \beta = 1 - \alpha\) we have therefore \(f_{R_i}(r) = \alpha p_G(r; 1, \sigma^2) + \beta p_G(r; -1, \sigma^2)\). For the r.v. \(\Psi_i = R_i c_i\), from basic probability theory we get

\[
f_{\psi_i|H_0}(\psi) = \alpha \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(\psi - \mu_c)^2}{2\sigma^2}} + \beta \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(\psi + \mu_c)^2}{2\sigma^2}},
\]

(31)

and \(\Pr\{\Psi_i \geq 0|H_0\} = K_0 \triangleq \alpha + (1/2 - \alpha) \text{erfc} \left(-c_i/\sqrt{2\sigma^2}\right)\).

Substituting in (29) and by using the identity [19]

\[
\int_{0}^{\infty} e^{-(a^2 + 2bt + c)} dt = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{a}} \text{erfc} \left(\frac{c}{\sqrt{a}}\right)
\]

(32)

with the real part of \(a\) greater than 0, we can express the c.h.f. in closed form as:

\[
\Phi_{V_i|H_0}(\nu) = \frac{1}{2} e^{-4\pi\nu(c_i + 2\pi\sigma^2)} \left[ \alpha \text{erfc} \left(\frac{c_i - j4\pi\nu\sigma^2}{\sqrt{2\sigma}}\right) + \beta e^{j2\pi\nu} \text{erfc} \left(\frac{-c_i - j4\pi\nu\sigma^2}{\sqrt{2\sigma}}\right) \right] + K_0.
\]

(33)

Note that for equiprobable data symbols \(\alpha = \beta = K_0 = 1/2\) and, as expected due to the symmetries of the r.v.’s involved, (33) is the same for \(c_i = 1\) and \(c_i = -1\). In other words, for \(H_0\) and equiprobable data symbols the particular SW pattern does not play any role. On the other side, SW patterns must be properly designed for the mixed data case, as previously discussed.
4.2.2. Case $\mathcal{H}_1$

We here consider the case where the sync word is present. Under this hypothesis, the generic $R_i$ is Gaussian distributed, with mean $c_i$ and variance $\sigma^2$. Thus, whatever $c_i$ is, the r.v. $\Psi_i = R_i c_i$ is Gaussian with mean value $c_i^2 = 1$, i.e.

$$ f_{\Psi_i | \mathcal{H}_1}(\psi) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(\psi - 1)^2}{2\sigma^2}}, \quad (34) $$

and consequently

$$ \Pr\{\Psi_i \geq 0 | \mathcal{H}_1\} = \frac{1}{2} \text{erfc} \left( -\frac{1}{\sqrt{2\sigma}} \right). $$

Hence, from (29) and (34) we get

$$ \Phi_{V_i | \mathcal{H}_1}(\nu) = \frac{1}{2} \text{erfc} \left( -\frac{1}{\sqrt{2\sigma}} \right) + \frac{1}{2} e^{-4\pi\nu(j+2\pi\sigma^2)} \text{erfc} \left( \frac{1 - j4\pi\nu\sigma^2}{\sqrt{2\sigma}} \right). \quad (35) $$

In conclusion, expressions (33) and (35) substituted in (30) give in closed form the ch.f. of the decision variable.

5. PERFORMANCE EVALUATION OF THE CORRELATION METRIC

5.1. Synchronizing with hard decisions (BSC channel)

In this section, for the purpose of comparison, the probabilities of missed detection and of emulation will be briefly reviewed assuming that only hard decisions (i.e. $\text{sign}(r_i)$) about the received samples $r_i$ are available, i.e., with a memoryless BSC channel assumption\[1], \[3].

According to the hypothesis testing theory, for BSC the optimal test consists in the comparison of the correlation $\Gamma(r) = \sum_{i=1}^{N} c_i \text{sign}(r_i)$ with a prefixed threshold $\lambda$

$$ \Gamma(r) = \sum_{i=1}^{N} c_i \text{sign}(r_i) \quad \mathcal{D}_1 \succ \mathcal{D}_0 \quad \lambda. \quad (36) $$

In this test, we choose to decide for $\mathcal{D}_1$ in case of equality. We remark that $\Gamma(R)$ is a discrete r.v., and consequently, with the above defined test only certain values of $P_{EM}, P_{MD}$ are possible. A continuous variation could be obtained by introducing a randomized decision rule \[14].

The probability of emulation is the probability that the correlation with a random sequence in the corrupted data bitstream exceeds the prefixed threshold $\lambda$

$$ P_{EM} = \Pr \{ \Gamma(R) \geq \lambda \mid \mathcal{H}_0 \}. \quad (37) $$

The p.d.f. of the correlation in (36) under $\mathcal{H}_0$ is

$$ f_{\Gamma}(\gamma) = \left(\frac{1}{2}\right)^N \sum_{i=0}^{N} \binom{N}{i} \delta(\gamma - N + 2i), \quad (38) $$

where $\left(\frac{1}{2}\right)^N$ is the probability of the generic $n$-tuple, $\binom{N}{i}$ gives the number of sequences with $i$ symbols coincident with that of the SW and $\delta(.)$ is the Dirac’s function.

Summing from the threshold value $\lambda$ to $N$, we obtain the probability that the bitstream can emulate a start code as

$$ P_{EM} = \left(\frac{1}{2}\right)^N \sum_{k=0}^{N-1} \binom{N}{k}, \quad (39) $$

\[3\]As usual we assume a BSC error probability $p < 1/2$.\[4\]
which clearly depends on the synchronization word length $N$ and on the predetermined threshold $\lambda$.

The probability of missed detection can be evaluated starting from the probability that more than a given number of errors occur in $N$ symbols, given $H_1$. If $p$ is the BSC error probability, we simply have

$$P_{MD} = \sum_{k=\lceil \frac{N-\lambda}{2} \rceil +1}^{N} \binom{N}{k} p^k (1-p)^{N-k}. \quad (40)$$

We observe that $P_{MD}$ depends on channel conditions, whereas $P_{EM}$ is not influenced by the channel. In fact, if SW’s are more damaged by channel errors their detection is harder. Lowering the threshold considered can help the detection, of course at the expense of an increased probability of emulation.

5.2. Synchronizing with soft values using the correlation

We assume here that soft values are provided to the synchronizer, and that the synchronization algorithm is still based on the comparison of the (soft) correlation $\Gamma(r) = \sum_{i=1}^{N} c_i r_i$ with a threshold $\lambda$

$$\Gamma(r) = \sum_{i=1}^{N} c_i r_i \overset{\mathcal{D}_1}{\geq} \lambda. \quad (41)$$

Although this choice could appear reasonable, there is no theoretical justification for its use when soft values are available, and for its superiority with respect to the test based on hard values (36). Indeed, in some cases the soft correlation based test is worse than that based on hard correlation.

For performance evaluation we start by writing the correlation given $H_0$ as

$$\Gamma(r) = \sum_{i=1}^{N} d_i c_i + \sum_{i=1}^{N} n_i c_i. \quad (42)$$

This can be seen as the sum of two r.v.’s; the first one is given by the sum of $N$ binary r.v.’s, while the second is a Gaussian r.v. given by the sum of $N$ Gaussian r.v.’s.

After some algebra we get [17]

$$P_{EM} = \left(\frac{1}{2}\right)^{N+1} \sum_{k=0}^{N} \binom{N}{k} \text{erfc} \left(\frac{\lambda - N + 2k}{\sqrt{2N} \sigma}\right). \quad (43)$$

On the other hand, the correlation given $H_1$ can be written as

$$\Gamma(r) = \sum_{i=1}^{N} c_i^2 + \sum_{i=1}^{N} n_i c_i = N + \sum_{i=1}^{N} n_i c_i. \quad (44)$$

It is thus

$$\text{Pr}\{\Gamma(R) < \lambda|H_0\} = \text{Pr}\left\{\sum_{i=1}^{N} n_i c_i < \lambda - N\right\}. \quad (45)$$

The summation term is a Gaussian r.v., being the sum of $N$ independent Gaussian r.v.’s, each with zero mean and variance $\sigma^2$. It is thus easy to evaluate the probability of missed detection as

$$P_{MD} = \frac{1}{2} \text{erfc} \left(\frac{N - \lambda}{\sqrt{2N} \sigma}\right). \quad (46)$$

It is worthwhile to remark that, regardless of the common engineering practice of preferring soft correlation, synchronizers based on hard correlation can provide better results for some channel conditions.
6. NUMERICAL RESULTS

We report in this Section some examples of numerical results obtained with the analysis above. In particular, we compare the likelihood ratio test with its generalized version and with correlation based tests.

As an example of application of the presented analysis for GLRT, we report in Fig. 3 and Fig. 4 the distributions of the metric under hypothesis $H_0$, for the probability of emulation, and under hypothesis $H_1$, for the probability of correct detection $P_D = 1 - P_{MD}$. Results refer to AWGN channel with $\sigma^2 = -3dB$. The case of unbalanced data is considered. Fig. 3 reports the case of $\alpha = 0.4$ and $\beta = 1 - \alpha = 0.6$; Fig. 4 reports the case of $\alpha = 0.6$ and $\beta = 1 - \alpha = 0.4$. A sync word composed of 24 bits equal to 1 has been considered. We remark here that the performance depend in the unbalanced case on the sync word. These figures have been obtained according to results described in Section 4.

We may observe that the performance in the case $H_1$ ($P_{MD}$) is unaffected by the generalization. In the case $H_0$ the performance (in terms of $P_{EM}$) depends on the distribution of the data $d_i$. We may also observe that in this case the performance also depends on the considered sync word. Let us suppose for example that the sync word is made of symbols $c = c_i = 1$. In this case the probability of emulation is higher than that of the equiprobable case if $\alpha > 0.5$, lower otherwise. If $\alpha > 0.5$ it is more probable in fact that the metric is higher than the threshold. Conversely, considering a sync word whose symbols assumes the values $c = c_i = -1$, the probability of emulation is higher than that of the equiprobable case if $\alpha < 0.5$, lower otherwise.

In order to compare the proposed GLRT metric with the correlation metrics (both with hard and soft decisions), we show in Fig. 5 the ROC (receiver operating characteristic) curves, for the case $N = 24$ and $\sigma^2 = -3dB$. The probabilities of emulation (false alarm) $P_{EM}$ and of correct detection $P_D$ are represented in the axes, parameterized by the threshold values. The corresponding curves obtained with the correlation metric are reported for comparison, together with simulation points validating the analysis. We can observe that a high gain can be achieved respect to correlations with the derived metric, for any value of $P_D$.

Moreover, we report in the same figure simulation results for the "mixed data" case, where the test is applied to a vector composed of random data in the first (last) $j$ positions, followed (preceded) by the shifted versions of the SW for the remaining $N - j$ positions, with $j = 1, \ldots, N - 1$. The SW pattern chosen for the simulation in the "mixed data" case is the one in [15]. These simulation points show that "mixed data" is often better than pure random data, justifying the two hypotheses approach used in the analysis above.

Fig. 6 shows the ROC curves for the case $N = 32$ and $\sigma^2 = 2dB$. The probabilities of emulation (false alarm) $P_{EM}$ and of correct detection $P_D$ are represented in the axes, parameterized by the threshold values. The corresponding curves obtained with the correlation metric are reported for comparison. We may observe that a high gain may be achieved with the derived LRT metric, for any value of the threshold $\lambda$, with respect to correlation based metrics. In the comparison between the results in the LRT and GLRT case, we have observed that their performance are almost indistinguishable, except for very bad channel conditions. For this reason, the figure refers to this latter case.

In order to show a more comprehensive comparison among the different tests, in Fig. 7 we report the probability of detection versus signal-to-noise ratio ($E_s/N_0 = 1/2\sigma^2$) for false sync word detection probabilities $P_{EM} = 10^{-5}$ and $P_{EM} = 10^{-3}$. It is confirmed that the GLRT greatly outperforms soft correlation. For example, for $P_{EM} = 10^{-3}$ and a target $P_D = 0.9$, the synchronizer using soft correlation as metric requires, in terms of $E_s/N_0$, about 3 dB more than GLRT.

7. CONCLUSIONS

Sequential frame synchronization for aperiodically embedded synchronization words has been afforded in the paper. According to the hypothesis testing approach and the likelihood ratio test, also in its generalized version, the relevant detection metrics have been derived. An analytical performance evaluation of the synchronizers based on LRT and GLRT metrics has then been given and represented in terms of ROC curves. The general case of non-equally distributed data symbols has been considered in the latter case. In order to have a benchmark for the performance of the new metric, the performance of the correlation metrics has also been considered and represented in terms of ROC curves. A high gain is obtained with both the derived metrics with respect to
commonly used correlation based frame synchronizers. Since the GLRT metric is simpler to implement and looses very little in performance respect to optimal LRT for practical signal to noise ranges, GLRT based synchronizers could be preferred in most cases. For very low signal to noise ratios the gap between the two becomes more evident and the resort to the LRT may give substantial performance improvements.

Acknowledgment

This work was supported by the European Commission under project FP6 IST-001812 "PHOENIX".

REFERENCES

**Figure 1.** Frame Structure.

![Frame StructureDiagram](image)

**Figure 2.** Optimal sequential synchronizer scheme.

![Optimal Sequential Synchronizer Scheme](image)

**Figure 3.** Probability density function (left) and cumulative distribution function (right) of the GLRT metric. $\alpha = 0.4$, $E_s/N_0 = 0dB$, $N = 24$ bits.

![Probability Density Function and Cumulative Distribution Function](image)
Figure 4. Probability density function (left) and cumulative distribution function (right) of the GLRT metric. $\alpha = 0.6$, $E_s/N_0 = 0dB$, $N = 24$ bits.

Figure 5. Receiver Operating Characteristic (ROC) curves. Sync word of $N = 24$ bits, $\sigma^2 = -3dB$. 
Figure 6. Receiver Operating Characteristic (ROC) curves. Sync word of $N = 32$ bits, $\sigma^2 = 2dB$.

Figure 7. Probability of detection versus signal-to-noise ratio for false sync word detection probability $P_{EM} = 10^{-5}, 10^{-3}$. Sync word of $N = 32$ bits.